

MAS212 Linear Algebra I Exercises 8 (with solutions)

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Assessed submission deadline 5pm on Friday 1st December 2006 in the blue box on the ground floor.

Your submission must have your FULL NAME and STUDENT NUMBER PRINTED CLEARLY at the top and separate sheets must be stapled together.

Marks out of 100 are shown in square brackets after each question number.

1. [30] Let V be a vector space. The *identity map* on V , $Id_V: V \rightarrow V$, is defined by $Id_V(v) = v$ for all $v \in V$. If $\mathcal{B}, \mathcal{B}'$ are two ordered bases for V , determine the change-of-basis matrix $P = (Id_V; \mathcal{B}, \mathcal{B}')$

and its inverse P^{-1} when $V, \mathcal{B}, \mathcal{B}'$ are as follows:

(a) $V = \mathbb{R}^3$, $\mathcal{B} = \mathcal{E}_3$ (the standard ordered basis on $\mathbb{R}^3$), $\mathcal{B}' = (1, 0, 0), (1, 1, 0), (1, 1, 1)$;

(b) $V = \mathbb{R}^4$, $\mathcal{B} = (-1, 1, 1, 0), (1, -1, 0, 1), (1, 0, -1, 1), (0, 1, 1, -1)$, $\mathcal{B}' = \mathcal{E}_4$ (the standard ordered basis on $\mathbb{R}^4$).

Solution

(a) Construct the change-of-basis matrix P from $\mathcal{B}$ to $\mathcal{B}'$. Solving the vector equation

$Id(1, 0, 0) = (1, 0, 0) = a(1, 0, 0) + b(1, 1, 0) + c(1, 1, 1)$ for the scalar coefficients a, b, c gives

$Id(1, 0, 0) = (1, 0, 0) = 1(1, 0, 0) + 0(1, 1, 0) + 0(1, 1, 1)$, and similarly

$Id(0, 1, 0) = (0, 1, 0) = -1(1, 0, 0) + 1(1, 1, 0) + 0(1, 1, 1)$, and

$Id(0, 0, 1) = (0, 0, 1) = 0(1, 0, 0) - 1(1, 1, 0) + 1(1, 1, 1)$, hence

$$P := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}. \text{ To find } P^{-1} \text{ use matrix inversion to give } P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \text{ Check: } P P^{-1} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Or construct the change-of-basis matrix P^{-1} from $\mathcal{B}'$ to $\mathcal{B}$. Solving the vector equation

$Id(1, 0, 0) = (1, 0, 0) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$ for the scalar coefficients a, b, c gives

$Id(1, 0, 0) = (1, 0, 0) = 1(1, 0, 0) + 0(0, 1, 0) + 0(0, 0, 1)$, and similarly

$Id(1, 1, 0) = (1, 1, 0) = 1(1, 0, 0) + 1(0, 1, 0) + 0(0, 0, 1)$, and

$Id(1, 1, 1) = (1, 1, 1) = 1(1, 0, 0) + 1(0, 1, 0) + 1(0, 0, 1)$, hence

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \text{ Note that because the target basis } \mathcal{B} \text{ is the standard basis, the columns of the}$$

change-of-basis matrix P^{-1} are just the vectors in the source basis $\mathcal{B}'$. This must always be the case.

(b) Construct the change-of-basis matrix P from $\mathcal{B}$ to $\mathcal{B}'$.

$Id(-1, 1, 1, 0) = (-1, 1, 1, 0) = -1(1, 0, 0, 0) + 1(0, 1, 0, 0) + 1(0, 0, 1, 0) + 0(0, 0, 0, 1)$, etc. Hence, because the target basis $\mathcal{B}'$ is the standard basis, the columns of the change-of-basis matrix P are the elements of $\mathcal{B}$, i.e.

$$P := \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}. \text{ To find } P^{-1} \text{ either use matrix inversion to give } P^{-1} =$$

$$\begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \text{ or proceed as in part (a).}$$

(One way to compute P^{-1} is first to compute the matrix of minors of P , using (say) diagonal expansion for each minor, to give

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & 2 & -1 \\ -1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 1 \end{bmatrix}$$

from which the adjoint of P is

$$\begin{bmatrix} 1 & -1 & -1 & -2 \\ -1 & 1 & -2 & -1 \\ -1 & -2 & 1 & -1 \\ -2 & -1 & -1 & 1 \end{bmatrix}$$

Then Laplace expansion about (say) the first row gives the determinant of P as -3 and transposing the adjoint and dividing by the determinant gives the inverse as above.)

2. [20] The linear map $\alpha : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is represented by the matrix $A = (\alpha; \mathcal{E}_4, \mathcal{E}_3) = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix}$,

where $\mathcal{E}_3, \mathcal{E}_4$ are the standard ordered bases on $\mathbb{R}^3, \mathbb{R}^4$. Using information from your solution to

Question 1 above where appropriate (or otherwise) write down or calculate the following matrices:

(a) $P = (Id_{\mathbb{R}^4}, \mathcal{E}_4, \mathcal{C})$ and P^{-1} , where $\mathcal{C} = (-1, 1, 1, 0), (1, -1, 0, 1), (1, 0, -1, 1), (0, 1, 1, -1)$;

(b) $Q = (Id_{\mathbb{R}^3}, \mathcal{E}_3, \mathcal{D})$, where $\mathcal{D} = (1, 0, 0), (1, 1, 0), (1, 1, 1)$;

(c) $A' = (\alpha; \mathcal{C}, \mathcal{D}) = Q A P^{-1}$.

Solution

(a) P here is P^{-1} in Question 1(b), so $P = \frac{1}{3} \begin{bmatrix} -1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & -1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

(b) Q here is P in Question 1(a), so $Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) Hence $A' = Q A P^{-1} =$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & -1 & 5 \\ -2 & -2 & -2 & -2 \\ 3 & 4 & 1 & 2 \end{bmatrix}.$$

3. [50] Regard the matrix A in Question 2 above as a linear map $\alpha: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ given by $y = A x$ where x, y are column vectors.

(a) Find a basis (if one exists) for $\ker(A)$ and extend it (on the left) to an ordered basis for the domain ($\mathbb{R}^4$).

(b) Choose a basis for $\text{im}(A)$ from the image of the basis for the domain (excluding the basis for the kernel) and, if necessary, extend it (on the right) to a basis for the codomain ($\mathbb{R}^3$).

(c) Obtain an invertible 4×4 matrix P and an invertible 3×3 matrix Q such that $Q A P$ has the

general form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$, where I_r is the $r \times r$ identity matrix, $r = \text{rank}(A)$, and the zero submatrices

represented by O may have different shapes and some may be absent.

Solution

(a) Solving $A x = 0$ gives the following solution $\begin{bmatrix} -2t \\ t \\ 0 \\ t \end{bmatrix}$, where t is an arbitrary real number,

which represents the kernel as a one-dimensional vector subspace. Hence, a convenient choice of

basis (by choosing $t = 1$) is $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. We can then extend this basis on the left to give

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ as an ordered basis for the domain ($\mathbb{R}^4$).

(b) Mapping only the extension of the basis for the domain (thereby avoiding the kernel) gives a

spanning set for the image as $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$. These 3 vectors are obviously linearly independent and so must form a basis for the whole codomain ($\mathbb{R}^3$).

(c) The change-of-basis matrix in the domain from the new basis to the standard basis is the

matrix whose columns are the new basis vectors, i.e. $P = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Similarly, the change-

of-basis matrix in the codomain from the new basis to the standard basis is the matrix whose

columns are the new basis vectors, i.e. $Q^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 4 \end{bmatrix}$. Inverting gives $Q = \begin{bmatrix} -\frac{3}{4} & -\frac{5}{8} & \frac{7}{8} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \end{bmatrix}$

and hence $QAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. **Optional remark:** the rank of A is 3.